

## Evaluating measures of nonclassical correlation in a multipartite quantum system

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We introduce and compare several measures of nonclassical correlation defined on the basis of a widely-recognized paradigm claiming that a multipartite system represented by a density matrix having no product eigenbasis possesses nonclassical correlation.

*Keywords:* Nonclassical correlation; Numerical method.

### 1. Introduction

Classical/nonclassical separation is a controversial subject. There have been debates on the definition of nonclassical correlation of a multipartite system. It is well-known that the separability paradigm<sup>1,2,3</sup> is widely-accepted to define entanglement as a quantum correlation that cannot be generated from scratch using only local operations and classical communication (LOCC) (See, e.g., Ref. 4). There are different paradigms to define nonclassical correlation of a multipartite system from operational viewpoints. Bennett *et al.*<sup>5</sup> discussed a certain nonlocality about locally nonmeasurable separable states. Ollivier and Zurek<sup>6</sup> introduced a measure called quantum discord defined as a discrepancy of two expressions of a mutual information that should be equivalent to each other in a classical information theory. A simple classical/nonclassical separation was given by Oppenheim *et al.*<sup>7,8</sup> which is also widely recognized. They defined the class of (properly) classically correlated states that are the states with a product eigenbasis. Its complement is the class of nonclassically correlated states that are the states without product eigenbasis. This definition is in accord with their measure called quantum deficit defined as a discrepancy between the information that can be localized by applying closed LOCC (CLOCC) operations and the total information of the system. The CLOCC protocol allows only local unitary operations, attaching ancillas in separable pure states, and operations to send subsystems through a complete dephasing channel. Thus classically correlated states have vanishing quantum deficit. Other measures<sup>9,10</sup>

were later proposed on the basis of the same definition of classical/nonclassical correlation.

We aim to evaluate measures based on the following separation of classical/nonclassical correlations.

**Definition 1 (Oppenheim-Horodecki).** A quantum bipartite system consisting of subsystems A and B is (properly) classically correlated if and only if it is described by a density matrix having a biproduct eigenbasis.

A straightforward extension gives the definition:

**Definition 2.** A quantum multipartite system consisting of subsystems  $1, \dots, m$  is nonclassically correlated if and only if it is described by a density matrix having no  $m$ -product eigenbasis.

The set of classically correlated states is a nonconvex subset of the set of separable states. Thus a convexity cannot be a property of a measure of nonclassical correlation. The natural statement is that a measure  $M$  of nonclassical correlation should satisfy the following conditions:

- (i)  $M = 0$  if a system is described by a density matrix having a product eigenbasis.
- (ii)  $M$  is invariant under local unitary operations.

These conditions are considered to be prerequisite hereafter.

In addition, the additivity property should be satisfied if one needs to compare systems with different dimensions. There are two ways to define additivity for a measure. One is defined as follows, which is valid for any measure of bipartite correlation.

**Definition 3.** Let  $F(\rho^{AB})_{A|B}$  be a measure of correlation between subsystems A and B of a bipartite system AB, where A|B denotes splitting between A and B. Then,  $F(\rho^{AB})_{A|B}$  is called an additive measure if and only if the equality  $F(\rho^{AB} \otimes \sigma^{CD})_{AC|BD} = F(\rho^{AB})_{A|B} + F(\sigma^{CD})_{C|D}$  holds.

Another definition of additivity may be introduced for a multipartite measure when we focus on scaling.

**Definition 4.** Let us denote a measure of  $m$ -partite nonclassical correlation by  $M_m(\sigma)$  where  $\sigma$  is a density matrix of an  $m$ -partite quantum system. First, the measure is fully additive if and only if  $M_{m_1 \times m_2}(\sigma_1 \otimes \sigma_2) = M_{m_1}(\sigma_1) + M_{m_2}(\sigma_2)$  with  $\sigma_1$  the density matrix of an  $m_1$ -partite system and  $\sigma_2$  the density matrix of an  $m_2$ -partite system. Second, the measure possesses weak additivity if and only if  $M_{m^n}(\sigma^{\otimes n}) = nM_m(\sigma)$ . Third, the measure possesses subadditivity if and only if  $M_{m_1 \times m_2}(\sigma_1 \otimes \sigma_2) \leq M_{m_1}(\sigma_1) + M_{m_2}(\sigma_2)$ .

In this contribution, we first compare the definitions and properties of measures of nonclassical correlation in Sec. 2. Then we compare these measures numerically in simple examples in Sec. 3. The paper concludes with several remarks.

## 2. Nonclassical Correlation Measures

Four measures of nonclassical correlation are evaluated for comparison together with the negativity (an entanglement measure) <sup>11</sup>.

The first one is an uncertainty remaining for a third party about the quantum state shared by multiple persons after the third party receives many reports from them <sup>10</sup>. These reports are measurement results on the shared state using local observables. Let us consider a density matrix  $\rho^{[1,\dots,m]}$  of an  $m$ -partite system. Consider local complete orthonormal bases  $\{|c_j^{[1]}\rangle\}_j, \dots, \{|c_x^{[m]}\rangle\}_x$ . Suppose the  $l$ th person locally makes measurements using the observable  $\sum_s s |c_s^{[l]}\rangle\langle c_s^{[l]}|$  and sends reports to the third party. Then, a measure of nonclassical correlation is given by

$$D(\rho^{[1,\dots,m]}) = \min_{\text{local bases}} \left( - \sum_{j,\dots,x} p_{j,\dots,x} \log_2 p_{j,\dots,x} \right) - S_{\text{vN}}(\rho^{[1,\dots,m]}) \quad (1)$$

with  $p_{j,\dots,x} = \langle c_j^{[1]} | \langle c_k^{[2]} | \dots \langle c_x^{[m]} | \rho^{[1,\dots,m]} | c_j^{[1]} \rangle | c_k^{[2]} \rangle \dots | c_x^{[m]} \rangle$  ( $S_{\text{vN}}$  is the von Neumann entropy). The value of  $D(\rho^{[1,\dots,m]})$  vanishes if  $\rho^{[1,\dots,m]}$  has a (fully) product eigenbasis. In addition,  $D(\rho^{[1,\dots,m]})$  is invariant under local unitary operations as is clear from its definition. Furthermore, it is fully additive in terms of Definition 4.

The second one is derived from a game of multiple persons sharing a quantum state. Let us consider an artificial game to find out eigenvalues of the reduced density matrix of a subsystem from eigenvalues of the density matrix of the total system. Suppose that Kate has the  $k$ th component of an  $m$ -partite quantum system. Let the dimension of the Hilbert space of the  $k$ th component be  $d^{[k]}$  and that of the Hilbert space of the total system be  $d_{\text{tot}}$ . She wants to know the eigenvalues  $\{e_j^{[k]}\}_{j=1}^{d^{[k]}}$  of the reduced density matrix of the  $k$ th component. Kate receives  $d_{\text{tot}}$  eigenvalues from Tony who knows all the eigenvalues of the total system. Kate partitions them into  $d^{[k]}$  sets. Summing up elements in individual sets, she has  $d^{[k]}$  mimic eigenvalues  $\{\tilde{e}_j^{[k]}\}$ . Thus a measure of nonclassical correlation for Kate can be

$$F_k(\rho^{[1,\dots,m]}) = \min_{\text{partitionings}} \left| \sum_{j=1}^{d^{[k]}} (\tilde{e}_j^{[k]} \log_2 \tilde{e}_j^{[k]} - e_j^{[k]} \log_2 e_j^{[k]}) \right|.$$

We may take the maximum over  $k$  to have the measure

$$G(\rho^{[1,\dots,m]}) = \max_k F_k(\rho^{[1,\dots,m]}).$$

This is equal to zero if  $\rho^{[1,\dots,m]}$  has a (fully) product eigenbasis. In addition, it is invariant under local unitary operations as is clear from its definition. It is subadditive in terms of Definition 4.

The third one is a sort of measures introduced by Groisman *et al.*<sup>9</sup>. This is defined in the following way. Consider a bipartite system (AB) represented by a density matrix  $\rho^{\text{AB}}$ . Then, (i) Find a basis that diagonalizes the state  $\text{Tr}_B \rho^{\text{AB}} \otimes \text{Tr}_A \rho^{\text{AB}}$ . (ii) Write  $\rho^{\text{AB}}$  under the basis found by (i) and delete all off-diagonal

elements. Denote this state as  $\rho'$ . (iii) The measure is calculated by a certain distance between  $\rho^{\text{AB}}$  and  $\rho'$ . We can take the difference in von Neumann entropy as the distance function to define the measure:

$$D_G(\rho^{\text{AB}}) = S_{\text{vN}}(\rho') - S_{\text{vN}}(\rho^{\text{AB}}).$$

This measure can be seen as a variant of measure  $D$  by taking a fixed set of local bases instead of searching for the minimum in (1). An extension to the multipartite case is obvious. This measure satisfies additivity in terms of Definition 3.

To define the fourth one, we can get a clue to construct a measure of nonclassical correlation from a conventional entanglement measure in the following way. A well known measure of entanglement is negativity<sup>11</sup>,  $N(\rho^{\text{AB}})$ , defined as the absolute value of the sum of negative eigenvalues of  $(I^A \otimes \Lambda_T^B)\rho^{\text{AB}}$  where  $\Lambda_T$  is the transposition. This can be in fact regarded as a measure of nonclassical correlation. Nevertheless, this obviously does not quantify nonclassical correlation of systems described by separable density matrices. Instead of negativity, one can define another measure using the partial transposition:

$$K(\rho^{\text{AB}}) = \sum_x |e_x - \tilde{e}_x|,$$

where  $e_x$  are the eigenvalues of  $\rho^{\text{AB}}$  and  $\tilde{e}_x$  are the eigenvalues of  $(I^A \otimes \Lambda_T^B)\rho^{\text{AB}}$ ; both  $e_x$ 's and  $\tilde{e}_x$ 's are aligned in the descending (or ascending) order. This measure utilizes the fact that  $\Lambda_T$  is eigenvalue-preserving while  $I \otimes \Lambda_T$  is, in general, not. The partial transposition  $I \otimes \Lambda_T$  preserves eigenvalues when acting on a density matrix having a biproduct eigenbasis. In this sense, any eigenvalue-preserving-but-not-completely-eigenvalue-preserving map (EnCE, this might be reminiscent of PnCP) can be used to define a measure of nonclassical correlation. This will be investigated in detail in our forthcoming contribution<sup>12</sup>. One drawback is that a measure in the form of  $K$  does not possess an additivity property, similarly to the case of negativity. The extension of this measure to a multipartite case can be accomplished by taking the minimum over all bipartite splittings.

### 3. Examples

We compare the four measures we have seen above together with negativity in three simple examples of bipartite splitting cases. The measure  $D$  requires a random search of local bases to estimate its value for a given state. We try  $4.0 \times 10^4$  randomly generated bases for each data point in Figs. 1 (b) and (c). Other measures can be calculated without numerical estimation.

The first example is the pseudo-entangled state of two qubits,  $\rho_{\text{ps}} = p|\psi\rangle\langle\psi| + (1-p)I/4$  with  $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ . For this state, we have  $D(\rho_{\text{ps}}) = D_G(\rho_{\text{ps}}) = 2s[(1+p)/4] - s[(1-p)/4] - s[(1+3p)/4]$  where  $s(x) = -x \log_2 x$  ( $0 \leq x \leq 1$ ). In addition, we have  $G(\rho_{\text{ps}}) = 1 - H[(1+p)/2]$  where  $H(x) = s(x) + s(1-x)$  is the binary entropy function. It is also easy to obtain  $N(\rho_{\text{ps}}) = |\min[0, (1-3p)/4]|$  and  $K(\rho_{\text{ps}}) = 2p$ . These results are plotted against  $p$  in Fig. 1 (a).

The second example is the two-qubit density matrix  $\sigma = (1/2 - p)(|00\rangle\langle 00| + |11\rangle\langle 11|) + 2p|\phi\rangle\langle\phi|$  with  $|\phi\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$  and  $0 \leq p \leq 1/2$ . This is inseparable for  $p > 1/4$  since  $(I \otimes \Lambda_T)\sigma$  has the eigenvalues  $1/2 - 2p$ ,  $p$  (with the multiplicity of two), and  $1/2$ . We need to estimate  $D(\sigma)$  using a numerical search. As for other measures, we obtain  $G(\sigma) = \min\{1 - H(p+1/2), 1 - H(2p)\}$ ,  $D_G(\sigma) = 2s(p) - s(2p)$ ,  $N(\sigma) = |\min[0, 1/2 - 2p]|$ , and  $K(\sigma) = 4p$  ( $0 \leq p \leq 1/6$ ),  $2 - 8p$  ( $1/6 < p \leq 1/4$ ),  $8p - 2$  ( $1/4 < p \leq 1/2$ ). These functions are illustrated in Fig. 1 (b) as functions of  $p$ .

The third example is the  $8 \times 8$  density matrix  $\sigma_b$  of the bipartite system  $AB$  with the dimensions of the Hilbert spaces, two for  $A$  and four for  $B$ .

$$\sigma_b = \frac{1}{7b+1} \left[ \text{diag}(b, b, b, b, \frac{1+b}{2}, b, b, \frac{1+b}{2}) + b \times (|0\rangle\langle 5| + |1\rangle\langle 6| + |2\rangle\langle 7| + |5\rangle\langle 0| + |6\rangle\langle 1| + |7\rangle\langle 2|) + \frac{\sqrt{1-b^2}}{2} \times (|4\rangle\langle 7| + |7\rangle\langle 4|) \right]$$

with  $0 \leq b \leq 1$ . This was originally introduced by Horodecki<sup>13</sup> as an entangled state with positive partial transpose. In fact  $\mathcal{N}(\sigma_b) = 0$  although it is inseparable for  $0 < b < 1$ . It is notable that  $I^A \otimes \Lambda_T^B$  does not change the eigenvalues of  $\sigma_b$  and hence  $N(\sigma_b) = K(\sigma_b) = 0$ . The values of other measures are numerically estimated or analytically calculated in a straightforward manner while these are too lengthy to include in the text. Plots of the measures against  $b$  are shown in Fig. 1 (c).

#### 4. Concluding Remarks

There are many different ways to define a measure of nonclassical correlation. We have seen four of them. The measures  $D$  and  $D_G$  look stable and faithful against changes of parameters among tested measures as far as we could see in the three simple examples. Further investigation is required to find desirable properties in addition to additivity properties. Computational cost should be another factor to choose a measure. A seemingly natural measure  $D$  cannot be used for a system with a large dimension due to the cost of searching over all possible local bases. Thus the measures other than  $D$  are good candidates in this sense.

We have found that a more general framework to detect and quantify nonclassical correlation can be constructed with EnCE, in analogy with the PnCP map theory, in the process of defining the measure  $K$ . Consider a map  $\Lambda$  such that  $\Lambda$  preserves the eigenvalues of a density matrix while  $I \otimes \Lambda$  in general does not. It is obvious that  $I \otimes \Lambda$  preserves eigenvalues of a density matrix if it has a biproduct eigenbasis. Thus such  $\Lambda$  can be used to detect and quantify nonclassical correlation. Further investigation in this approach will be reported elsewhere<sup>12</sup>.

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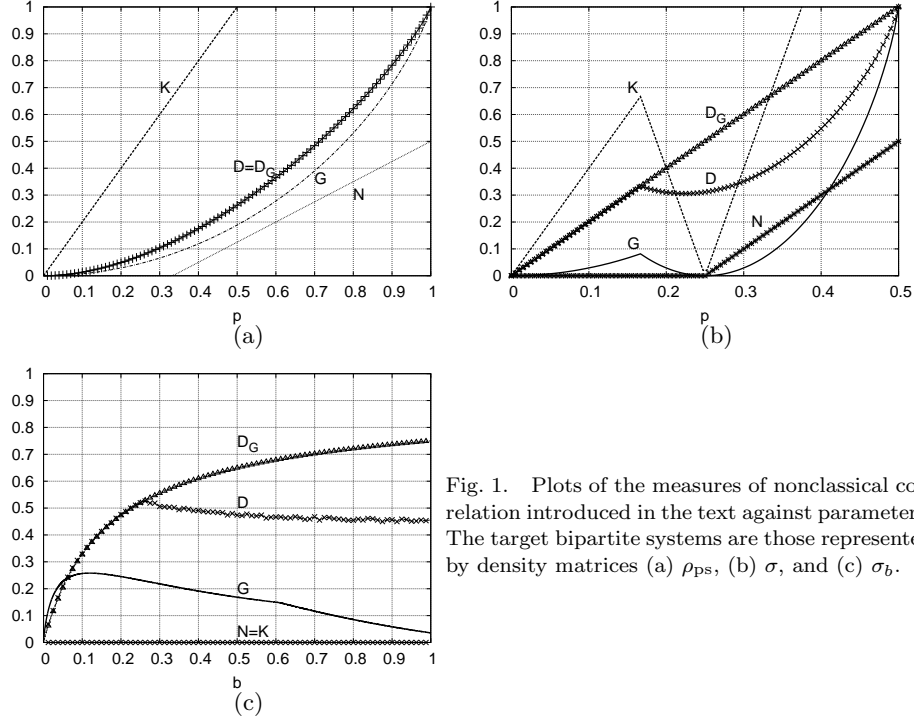


Fig. 1. Plots of the measures of nonclassical correlation introduced in the text against parameters. The target bipartite systems are those represented by density matrices (a)  $\rho_{ps}$ , (b)  $\sigma$ , and (c)  $\sigma_b$ .

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